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On the Weierstrass–Stone Theorem

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Let S be a compact Hausdorff space, and let E be a normed space over the reals. Let $C(S; E)$ be the linear space of all E-valued continuous functions f on S with the uniform norm

$$\|f\| = \sup\{\|f(t)\|; t \in S\}.$$

When $E = \mathbb{R}$, the Weierstrass–Stone Theorem describes the uniform closure of a subalgebra of $C(S; \mathbb{R})$. We extend this classical result in two ways: we admit vector-valued functions and describe the uniform closure of arbitrary subsets of $C(S; E)$. The classical Weierstrass–Stone Theorem is obtained as a corollary, without Zorn's Lemma.

INTRODUCTION

Let S be a compact Hausdorff space, and let $C(S; \mathbb{R})$ be the Banach space of all continuous real-valued functions f on S equipped with the sup-norm

$$\|f\| = \sup\{|f(t)|; t \in S\}.$$

Let $A \subset C(S; \mathbb{R})$ be a *subalgebra*, i.e., a subset such that $f + g$, fg, and λf belong to A, for all f, $g \in A$ and $\lambda \in \mathbb{R}$. Let us assume that A contains the constant functions and is separating over S, i.e., given $x \neq y$ in S, there is some $f \in A$ such that $f(x) \neq f(y)$. The classical Weierstrass–Stone Theorem states that under this hypothesis the subalgebra A is dense in $C(S; \mathbb{R})$: that is, for every $f \in C(S; \mathbb{R})$ and every $\varepsilon > 0$, there is some $g \in A$ such that $\|f - g\| < \varepsilon$. One generalization of this result is the following: suppose $W \subset C(S; \mathbb{R})$ is a linear subspace such that $fw \in W$ for every $f \in A$ and $w \in W$, where A is a subalgebra of $C(S; \mathbb{R})$, i.e., W is an

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A-module. Assume that A is separating over S and, for each $x \in S$, there is some $w \in W$ such that $w(x) \neq 0$. Then W is dense in $C(S; \mathbb{R})$. (See Nachbin [9].) In 1976, S. Machado discovered an elementary proof of this result on A-modules. In fact, he showed that for any A-module $W \subset C(S; \mathbb{R})$ and for any $f \in C(S; \mathbb{R})$,

$$\mathrm{dist}(f; W) = \sup_{x \in S} \mathrm{dist}(f(x); W(x))$$

whenever A is separating over S. For his proof, see Prolla [11, pp. 4–10]. In fact, Machado proved a similar formula for the case that A is not separating and, using a transfinite argument, applied it to give a new proof of Bishop's generalized Weierstrass–Stone Theorem. (See Machado [8] or Burckel [2].) In 1984, T. J. Ransford [12] gave an extremely simple proof of Machado's result, using Zorn's Lemma. Using the notation of multipliers of a subset $W \subset C(S; \mathbb{R})$, that was introduced for convex cones by Feyel and De La Pradelle [6], Chao-Lin extended Ransford's argument to arbitrary subsets of $C(S; \mathbb{R})$. (See [3].)

The purpose of this paper is to revise Machado's proof of the Weierstrass–Stone Theorem to extend it to the case of arbitrary subsets $W \subset C(S; \mathbb{R})$. A comparison of our proof with Machado's original proof, which appears in Prolla [11, pp. 4–10], shows that very little had to be changed. What prompted us to write it, was a desire to have a proof of the Weierstrass–Stone Theorem for *arbitrary* subsets of $C(S; \mathbb{R})$, that would *not* use Zorn's Lemma. The main tools are two results due to R. I. Jewett [7]. The first one is a polynomial approximation result which is a special case of Weierstrass Theorem. However, its direct proof is very simple: it depends only on Bernoulli's inequality. (See Lemma 2 of Jewett [7].) The second one is the fact that the closure of the set of multipliers of W (see Definition 2 below) is a lattice. This also has an elementary proof, albeit much more complicated. (See Theorem 1 of Jewett [7].) With these two results, we "approximate" the characteristic functions of neighborhoods of points: see the crucial Lemma 3 below. It is the analogue of Lemma 1.3 [11, p. 4] in Machado's proof of the Weierstrass–Stone Theorem. See also the analogous Lemma 1 of Brosowski and Deutsch [1], which is the essential step of their elementary proof of the classical Weierstrass–Stone Theorem. Notice that their proof is also free from the use of Zorn's Lemma. Finally, we build a partition of unity, using multipliers. For this purpose we use the method employed by Rudin (see Theorem 2.13 of Rudin [13]): it allows us to use only two kinds of operations: given two multipliers φ and ψ, form $(1 - \varphi)$ and $\varphi\psi$. These two operations were singled out by von Neumann as the most natural when dealing with a Weierstrass–Stone Theorem for sets of elements of $C(S; [0, 1])$. (See von Neumann [14, pp. 93–94, 15] and Jewett [7].)

1. Definitions and Lemmas

Throughout this paper S is a non-empty compact Hausdorff space and E is a non-trivial real or complex normed space. $C(S; E)$ is the linear space of all continuous functions from S into E, equipped with the supremum norm

$$\|f\| = \sup\{\|f(x)\|;\ x \in S\}.$$

When $E = \mathbb{R}$, we denote by $C(S; [0, 1])$ the subset of $C(S; \mathbb{R})$ consisting of those functions from S into the unit interval $[0, 1] \subset \mathbb{R}$. Following Jewett [7] we state the following definition.

Definition 1. A non-empty subset $M \subset C(S; [0, 1])$ is said to have property V, if

(1) $\varphi \in M$ implies $1 - \varphi$ belongs to M;

(2) $\varphi \in M$ and $\psi \in M$ implies $\varphi\psi \in M$.

Following Feyel and De La Pradelle [6] and Chao-Lin [3], we state our next definition.

Definition 2. Let $W \subset C(S; E)$ be a non-empty subset. A function $\varphi \in C(S; [0, 1])$ is called a *multiplier* of W if $\varphi f + (1 - \varphi)g$ belongs to W, for every pair, f and g, of elements of W.

Let M be the set of all multipliers of W. Clearly, M satisfies condition (1) of Definition 1. The identity

$$(\varphi\psi)f + (1 - \varphi\psi)g = \varphi[\psi f + (1 - \psi)g] + (1 - \varphi)g,$$

shows that M satisfies condition (2) as well. Hence M has property V.

Definition 3. A subset $X \subset C(S; \mathbb{R})$ is a said *to separate the points of* S if, given any two distinct points, s and t, of S, there is a function $\varphi \in X$ such that $\varphi(s) \neq \varphi(t)$.

Our first two lemmas are taken from Jewett [7].

Lemma 1. *Let $0 < a < b < 1$ and $0 < \delta < 1/2$ be given. There exists a polynomial $p(x) = (1 - x^m)^n$, such that*

(1) $p(t) > 1 - \delta$, *for all* $0 \leq t \leq a$,

(2) $p(t) < \delta$, *for all* $b \leq t \leq 1$.

Proof. See Lemma 2 of Jewett [7]. ∎

Lemma 2. *If $M \subset C(S; [0, 1])$ has property V, and φ and ψ belong to M, then* $\max(\varphi, \psi)$ *belongs to the uniform closure of M.*

Proof. See Theorem 1 of Jewett [7]. Just notice that the uniform closure of M in $C(S;[0,1])$ has property V too. ∎

In our next lemma we "approximate" the characteristic functions of open neighborhoods. It is the analogue of Lemma 1.3 of Machado's proof of the Weierstrass–Stone Theorem (Lemma 1.3 of Prolla [11, p. 4], and of Lemma 1 of Brosowski and Deutsch [1].

LEMMA 3. *Let $M \subset C(S;[0,1])$ be a non-empty separating subset with property V. Let $x \in S$ and let N be an open neighborhood of x in S. There exists an open neighborhood U of x, contained in N, such that, for each $0 < \delta < 1/2$, there is $\varphi \in M$ such that*

(1) $\varphi(t) > 1 - \delta$, *for all* $t \in U$,

(2) $\varphi(t) < \delta$, *for all* $t \notin N$.

Proof. Let K be the complement of N. For each $t \in K$, there is $\varphi_t \in M$ such that $\varphi_t(t) < \varphi_t(x)$. Choose real numbers a and b such that $\varphi_t(t) < a < b < \varphi_t(x)$. By Lemma 1, there is a polynomial $p_t(x) = (1 - x^m)^n$ such that $p_t(x) < 1/4$ for $b \leq x \leq 1$, and $p_t(x) > 3/4$ for $0 \leq x \leq a$. Hence $p_t(\varphi_t(x)) < 1/4$ and $p_t(\varphi_t(t)) > 3/4$. Let $U(t) = \{s \in S;\ p_t(\varphi_t(s)) > 3/4\}$. Then $U(t)$ is an open neighborhood of t. By compactness, there are $t_1, \ldots, t_m \in K$ such that $K \subset U(t_1) \cup U(t_2) \cup \cdots \cup U(t_m)$. For each $i = 1, \ldots, m$ let $\varphi_i(s) = p_{t_i}(\varphi_{t_i}(s)$, $s \in S$. Clearly, $\varphi_i \in M$, for all $i = 1, \ldots, m$. Let $\psi(s) = \max(\varphi_1(s), \ldots, \varphi_m(s))$, $s \in S$.

By Lemma 2 the function ψ belongs to the uniform closure of M. Notice that $\psi(x) < 1/4$ and $\psi(t) > 3/4$, for all $t \in K$. Define $U = \{s \in S;\ \psi(s) < 1/4\}$. Clearly, U is an open neighborhood of x in S. We claim that U is contained in N. Indeed, if $t \notin N$, then $t \in K$, and therefore $\psi(t) > 3/4$. Thus $t \notin U$, for all $t \notin N$.

If $0 < \delta < 1/2$ is given, let p be a polynomial determined by Lemma 1, applied to $a = 1/4$, $b = 3/4$, and $\delta/2$. Let $\eta(s) = p(\psi(s))$, for $s \in S$. Since the uniform closure of M also has the property V, the function η belongs to the uniform closure of M. If $t \in U$, then $\eta(t) > 1 - \delta/2$. If $t \notin N$, then $t \in K$ and $\eta(t) < \delta/2$. Finally, choose $\varphi \in M$, such that $\|\varphi - \eta\| < \delta/2$. Then φ satisfies (1) and (2). ∎

2. THE WEIERSTRASS–STONE THEOREM

Here is our Weierstrass–Stone theorem for *arbitrary* non-empty subsets.

THEOREM 1. *Let W be a non-empty subset of $C(S; E)$ such that the set M of all multipliers of W separates the points of S. Let $f \in C(S; E)$ and*

$\varepsilon > 0$ be given. The following are equivalent:

(1) *there is some $g \in W$ such that $\|f - g\| < \varepsilon$;*

(2) *for each $x \in S$, there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < \varepsilon$.*

Proof. Clearly (1) $\Rightarrow$ (2). Conversely, assume (2) is true. For each $x \in S$, there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < \varepsilon$. Choose a real number $\varepsilon(x) > 0$ such that $\|f(x) - g_x(x)\| < \varepsilon(x) < \varepsilon$. Then

$$N(x) = \{t \in S; \|f(t) - g_x(t)\| < \varepsilon(x)\}$$

is an open neighborhood of x in S. Select a point $x_1 \in S$ arbitrarily. Let $K = S \setminus N(x_1)$. For each $x \in K$, select an open neighborhood $U(x)$ of x, contained in $N(x)$, by Lemma 3 applied to the set M of multipliers of W. By compactness of K, there exists a finite set $\{x_2, \ldots, x_m\} \subset K$ such that $K \subset U(x_2) \cup \cdots \cup U(x_m)$. Let $\varepsilon' = \max\{\varepsilon(x_i); 1 \le i \le m\}$. Clearly, $\varepsilon' < \varepsilon$. Let $k = \max\{\|f - g_1\|, \ldots, \|f - g_m\|\}$, where we have defined $g_i = g_{x_i}$ for $i = 1, 2, \ldots, m$. Choose now $0 < \delta < 1/2$ so small that $\delta k m < \varepsilon - \varepsilon'$. By Lemma 3, there are $\varphi_2, \ldots, \varphi_m \in M$ such that

$$\varphi_i(x) > 1 - \delta, \qquad \text{for all } x \in U(x_i), \tag{1}$$

$$\varphi_i(t) < \delta, \qquad \text{for all } t \notin N(x_i), \tag{2}$$

for all $i = 2, 3, \ldots, m$. Define

$$\begin{aligned} \psi_2 &= \varphi_2, \\ \psi_3 &= (1 - \varphi_2)\varphi_3, \\ &\cdots \\ \psi_m &= (1 - \varphi_2)(1 - \varphi_3) \cdots (1 - \varphi_{m-1})\varphi_m. \end{aligned}$$

Clearly, $\psi_i \in M$ for all $i = 2, 3, \ldots, m$. Now

$$\psi_2 + \cdots + \psi_j = 1 - (1 - \varphi_2) \cdots (1 - \psi_j), \qquad j = 2, \ldots, m,$$

can be easily verified by induction. Define

$$\psi_1 = (1 - \varphi_2)(1 - \varphi_3) \cdots (1 - \varphi_m).$$

Then $\psi_1 \in M$ and $\psi_1 + \psi_2 + \cdots + \psi_m = 1$. Notice that

$$\psi_i(t) < \delta, \qquad \text{for all } t \notin N(x_i),\ i = 1, 2, \ldots, m. \tag{3}$$

Indeed, if $i \ge 2$, then $\psi_i(t) \le \varphi_i(t)$ and (3) follows from (2). If $i = 1$, and $t \notin N(x_i)$, then $t \in K$. Hence $t \in U(x_j)$, for some $j = 2, \ldots, m$. By

(1), $1 - \varphi_j(t) < \delta$ and so

$$\psi_1(t) = \left(1 - \varphi_j(t)\right) \cdot \prod_{i \neq j} \left(1 - \varphi_i(t)\right) < \delta.$$

Let $g = \psi_1 g_1 + \psi_2 g_2 + \cdots + \psi_m g_m$. Notice that

$$g = \varphi_2 g_2 + (1 - \varphi_2)\left[\varphi_3 g_3 + (1 - \varphi_3) \times\left[\varphi_4 g_4 + \cdots + (1 - \varphi_{m-1}) \times\left[\varphi_m g_m + (1 - \varphi_m) g_1\right] \cdots \right]\right].$$

Hence $g \in W$. Let $x \in S$ be given. Define $I = \{1 \leq i \leq m;\ x \in N(x_i)\}$ and $J = \{1 \leq i \leq m;\ x \notin N(x_i)\}$. Then

$$\|f(x) - g(x)\| = \left\| \sum_{i=1}^{m} \psi_i(x)(f(x) - g_i(x)) \right\| \leq \sum_{i=1}^{m} \psi_i(x) \|f(x) - g_i(x)\|.$$

Now $i \in I$ implies $\|f(x) - g_i(x)\| < \varepsilon(x_i) \leq \varepsilon'$. Hence

$$\sum_{i \in I} \psi_i(x) \|f(x) - g_i(x)\| < \varepsilon' \sum_{i \in I} \psi_i(x) \leq \varepsilon'. \tag{4}$$

On the other hand, by (3), $i \in J$ implies $\psi_i(x) < \delta$. Hence

$$\sum_{i \in J} \psi_i(x) \|f(x) - g_i(x)\| < \delta k m < \varepsilon - \varepsilon'. \tag{5}$$

From (4) and (5) we get $\|f(x) - g(x)\| < \varepsilon$. ∎

COROLLARY 1. *Let W be a non-empty subset of $C(S; E)$ such that the set M of all multipliers of W separates the points of S. Let $f \in C(S; E)$ be given. The following are equivalent:*

(1) *f belongs to the uniform closure of W in $C(S; E)$;*

(2) *for each $x \in S$, the value $f(x)$ belongs to the closure of the set $W(x) = \{g(x);\ g \in W\}$ in E.*

Proof. This is immediate from Theorem 1. ∎

COROLLARY 2. *Let W be a non-empty subset of $C(S; E)$ such that*

(1) *for each pair of distinct points, x and y, of S there is some multiplier φ of W such that $\varphi(x) \neq \varphi(y)$;*

(2) *for each $x \in S$, $v \in E$, and $\varepsilon > 0$, there is some $g \in W$ such that $\|g(x) - v\| < \varepsilon$.*

Then W is dense in $C(S; E)$.

Proof. By (1), the set M of all multipliers of W is separating over S. By (2), the set $\{g(x);\ g \in W\}$ is dense in E. Hence, every $f \in C(S; E)$ verifies (2) of Corollary 1, and therefore belongs to the uniform closure of W in $C(S; E)$. ∎

For our next result we recall the definition of the distance of an element $f \in C(S; E)$ from W:

$$\operatorname{dist}(f; W) = \inf\{\|f - g\|;\ g \in W\}.$$

THEOREM 2. *Let W be a non-empty subset of $C(S; E)$ such that the set M of all multipliers of W separates the points of S. For each $f \in C(S; E)$ there exists $x \in S$ such that*

$$\operatorname{dist}(f; W) = \operatorname{dist}(f(x); W(x)).$$

Proof. If $\operatorname{dist}(f; W) = 0$, then $\operatorname{dist}(f(x); W(x)) = 0$ for every $x \in S$. Suppose $\operatorname{dist}(f; W) = d > 0$. By contradiction assume that for each point $x \in S$, $\operatorname{dist}(f(x); W(x)) < d$. Hence there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < d$. Consequently, f and $d > 0$ satisfy condition (2) of Theorem 1. By the equivalence between conditions (1) and (2) of Theorem 1, there is some $g \in W$ such that $\|f - g\| < d = \operatorname{dist}(f; W)$, a contradiction. ∎

Our next result is a Weierstrass–Stone theorem for *linear subspaces* of $C(S; E)$.

THEOREM 3. *Let $W \subset C(S; E)$ be a vector subspace such that*

$$A = \{\varphi \in C(S; \mathbb{R});\ \varphi g \in W, \text{ for all } g \in W\}$$

separates the point of S, and for each $x \in S$, $v \in E$, and $\varepsilon > 0$, there is some $g \in W$ such that $\|g(x) - v\| < \varepsilon$.

Then W is uniformly dense in $C(S; E)$.

Proof. Notice that the set A is a subalgebra of $C(S; \mathbb{R})$ containing the constants. The set $M = \{\varphi \in A;\ 0 \leq \varphi \leq 1\}$ is the set of all multipliers of W. Given $x \neq y$, by hypothesis there is some $\varphi \in A$ such that $\varphi(x) \neq \varphi(y)$. Since A is an algebra containing the constants, a standard argument shows that we may assume that $\varphi(x) = 0$ and $\varphi(y) = 1$. Let $\psi = \varphi^2/\|\varphi^2\|$. Then $\psi \in M$, and $\psi(x) = 0$, $\psi(y) = 1$. Hence M separates the points of S, and condition (1) of Corollary 2 is verified. By hypothesis, condition (2) of Corollary 2 is verified also. ∎

Remark. The idea of defining the algebra A of multipliers of a vector space $W \subset C(S; E)$ goes back to A. J. Ellis [5].

COROLLARY 3. *Let $W \subset C(S;\mathbb{R})$ be a vector subspace such that*

$$A = \{\varphi \in C(S;\mathbb{R});\ \varphi g \in W, \text{ for all } g \in W\}$$

separates the points of S. Assume that, for each $x \in S$ there is some $g \in W$ such that $g(x) \neq 0$.

Then W is uniformly dense in $C(S;\mathbb{R})$.

Proof. Let $x \in S$ and $v \in \mathbb{R}$ be given. Choose $g \in W$ such that $g(x) \neq 0$. Then $h(t) = (v/g(x))g(t)$, $t \in S$, belongs to W and $h(x) = v$. By Theorem 2, W is uniformly dense in $C(S;\mathbb{R})$. ∎

Our next result is the classical Weierstrass–Stone for *subalgebras* of $C(S;\mathbb{R})$.

THEOREM 4. *Let $W \subset C(S;\mathbb{R})$ be a subalgebra. Then W is uniformly dense in $C(S;\mathbb{R})$ if, and only if, the following two conditions hold*:

(1) *for every pair of distinct points, x and y, of S, there is $g \in W$ such that $g(x) \neq g(y)$,*

(2) *for every $x \in S$, there is some $g \in W$ such that $g(x) \neq 0$.*

Proof. The conditions are easily seen to be necessary. Conversely, assume that (1) and (2) hold. Let

$$A = \{\varphi \in C(S;\mathbb{R});\ \varphi g \in W, \text{ for all } g \in W\}.$$

Clearly, W being a subalgebra implies that $W \subset A$. By (1) the algebra W, and a fortiori A, separates the points of S. It remains to apply Corollary 3. ∎

3. APPLICATIONS

EXAMPLE 1. Let S be a compact Hausdorff space, and let $\varphi: S \to [0,1]$ be a continuous one-to-one mapping. Let W be the set of all functions f of the form

$$f(t) = \sum_{i+j\leq n} a_{ij}\varphi(t)^i(1-\varphi(t))^j, \qquad t \in S, \qquad (*)$$

where each a_{ij} is a *positive rational number*, and $n = 0, 1, 2, \dots$. Clearly, φ is a multiplier of W, and therefore the set M of all multipliers of W separates the points of S. On the other hand, $W(x) \supset Q_+$, the set of all positive rational numbers, for each $x \in S$. By Corollary 1, every function $f \in C(S;\mathbb{R})$ which is positive on S, i.e., $f(x) \geq 0$ for all $x \in S$, belongs to the uniform closure of W.

More generally, let W be the set of all functions f of the form (*), where the coefficients a_{ij} are restricted by some condition, say $a_{ij} \in A$, for all $i, j = 0, 1, 2, \ldots$, where A is some subset of $\mathbb{R}$ such that $A + A \subset A$. Let B be the closure of A in $\mathbb{R}$. By Corollary 1, every function $f \in C(S; \mathbb{R})$ which maps S into B, i.e., $f(S) \subset B$, belongs to the uniform closure of W. In the previous example, $A = Q_+$, and then $B = \mathbb{R}_+$.

A similar example is provided in the approximation of operator-valued functions. Let $\mathscr{B}(H)$ be the Banach space of all bounded linear operators on a Hilbert space H. Let $\mathscr{P}(H)$ be the convex cone of all *positive* operators, i.e., $\mathscr{P}(H)$ consists of those $T \in \mathscr{B}(H)$ such that $(Tv, v) \geq 0$, for every $v \in H$. Let W be the convex cone of all operator-valued continuous functions of the form

$$p(t) = \sum_{i+j \leq n} t^i (1-t)^j P_{ij}, \qquad 0 \leq t \leq 1,$$

where each $P_{ij} \in \mathscr{P}(H)$, $n = 0, 1, 2, \ldots$. Clearly, $\varphi(t) = t$ is a multiplier of W, and $W(x) = \mathscr{P}(H)$, for all $x \in [0,1]$. By Corollary 1, every $f \in C([0,1]; \mathscr{B}(H))$ such that $f(t) \in \mathscr{P}(H)$, for all $t \in [0,1]$, belongs to the uniform closure of W in $C([0,1]; \mathscr{B}(H))$. When the interval $[0,1]$ is substituted by an arbitrary compact Hausdorff space S, we take W to be the convex cone of all finite sums of functions of the form $t \in S \Rightarrow f(t)P$, where $f \in C(S; \mathbb{R}_+)$ and $P \in \mathscr{P}(H)$. Now, every $\varphi \in C(S; [0,1])$ is a multiplier of W, and $W(t) = \mathscr{P}(H)$, for every $t \in S$. By Corollary 1, every $f(t) \in C(S; \mathscr{B}(H))$ such that $f(t) \in \mathscr{P}(H)$, for all $t \in S$, belongs to the uniform closure of W in $C(S; B(H))$. Moreover, for every $f \in C(S; \mathscr{B}(H))$, by Theorem 2 there is some $x \in S$ such that

$$\operatorname{dist}(f; W) = \inf_{P \in \mathscr{P}(H),} \|f(x) - P\| = \operatorname{dist}(f(x); \mathscr{P}(H)).$$

Let us now consider the approximation by *compact* operators. Let $\mathscr{K}(H)$ be the linear subspace of all compact operators $T \in \mathscr{B}(H)$, and let $\mathscr{F}(H)$ be the linear space of all *finite-rank* linear operators $T \in \mathscr{B}(H)$. It is well known that the closure of $\mathscr{F}(H)$ in $\mathscr{B}(H)$ consists of $\mathscr{K}(H)$. Let $W = C(S; \mathbb{R}) \otimes \mathscr{F}(H)$ be the linear subspace of $C(S; \mathscr{B}(H))$ consisting of all finite sums of functions of the form $s \in S \mapsto f(s)T$, where $f \in C(S; \mathbb{R})$ and $T \in \mathscr{F}(H)$. Clearly, every $\varphi \in C(S; [0,1])$ is a multiplier of W, and $W(t) = \mathscr{F}(H)$, for all $t \in S$. By Corollary 1, every $f \in C(S; \mathscr{B}(H))$ such that $f(t) \in \mathscr{K}(H)$, for all $t \in S$, belongs to the uniform closure of $W = C(S; \mathbb{R}) \otimes \mathscr{F}(H)$. Moreover, by Theorem 2,

$$\operatorname{dist}(f; C(S; \mathscr{K}(H))) = \sup_{x \in S} \inf_{T \in \mathscr{K}(H)} \|f(x) - T\|$$

for every $f \in C(S; \mathscr{B}(H))$.

EXAMPLE 2. We can generalize the Weierstrass Theorem to any number of dimensions. Indeed, let S be a compact subset of some real normed space E (finite or infinite dimensional). For each $n = 1, 2, 3, \ldots,$ let $\mathscr{P}^n(E)$ denote the vector space of all continuous n-homogeneous real-valued polynomials of finite type on E. An element of $\mathscr{P}^n(E)$ is a finite sum of functions of the form

$$p(t) = \varphi_1(t) \cdots \varphi_n(t), \qquad t \in E,$$

where each φ_i is a continuous linear functional. Let $\mathscr{P}^0(E)$ be the vector space of all constant real-valued functions. Let $\mathscr{P}(E)$ be the vector space of all continuous polynomials of finite type on E. An element $p \in \mathscr{P}(E)$ is by definition of the form

$$p = \sum_{k=0}^{n} p_k$$

for some $n = 0, 1, 2, 3, \ldots,$ where each polynomial $p_k \in \mathscr{P}^k(E)$, $k = 0, 1, \ldots, n$. Let $A = \{\varphi \in C(S; \mathbb{R});\ \varphi p \in \mathscr{P}(E), \text{ for all } p \in \mathscr{P}(E)\}$. Clearly, if φ is the restriction to S of some element of the dual space E^*, then $\varphi \in A$. By the Hahn–Banach Theorem, A is then separating over S. Since $\mathscr{P}(E)$ contains all the constant real-valued functions, it follows from Theorem 2, that $\mathscr{P}(E)$ is dense in $C(S; \mathbb{R})$.

EXAMPLE 3. (Pelczynski [10]). Let S be a compact Hausdorff space and let E be a normed space over $\mathbb{R}$. Let $W \subset C(S; E)$ be a *polynomial algebra*, i.e., a vector subspace such that for each $g, h \in W$ and $\varphi^* \in E^*$ (E^* is the dual of E) the mapping $s \mapsto \varphi^*(g(s))f(s)$ belongs to W.

THEOREM 5. *For any polynomial algebra $W \subset C(S; E)$, the following are necessary and sufficient conditions for W to be dense in $C(S; E)$:*

(1) *for any pair of distinct points, x and y, of S there is some $g \in W$ such that $g(x) \neq g(y)$;*

(2) *for any $x \in S$, $v \in E$, and $\varepsilon > 0$, there is some $g \in W$ such that $\|g(x) - v\| < \varepsilon$.*

Proof. The conditions are easily seen to be necessary. Conversely assume (1) and (2) are true. Let

$$A = \{\varphi \in C(S; \mathbb{R});\ \varphi g \in W, \text{for all } g \in W\}.$$

If $x \neq y$ are given, choose $g \in W$ such that $g(x) \neq g(y)$. By the Hahn–Banach theorem, there is some $\varphi^* \in E^*$ such that $\varphi^*(g(x)) \neq \varphi^*(g(y))$. Let $\varphi(s) = \varphi^*(g(s))$ for all $s \in S$. Now, for each $h \in W$, the mapping $\varphi(s)h(s) = \varphi^*(g(s))h(s)$, $s \in S$, belongs to W. Hence $\varphi \in A$

and $\varphi(x) \neq \varphi(y)$, and so A is separating over S. It remains to apply Theorem 3. ∎

Notice that, if the polynomial algebra W also satisfies the condition that $s \mapsto \varphi^*(g(s))v$ belongs to W, for every $g \in W$, $v \in E$, and $\varphi^* \in E^*$, then (2) is equivalent to

(3) *for every $x \in S$, there is some $g \in W$ such that $g(x) \neq 0$.*

In particular, if $W \subset C(S; E)$ is a polynomial algebra that separates the points (condition (1)) and contains the constant mappings, then W is uniformly dense.

If S is a compact subset of a real normed space G and E is another normed space over $\mathbb{R}$, then $\mathscr{P}(G) \otimes E$ is such that its restriction to S is a polynomial algebra in $C(S; E)$ that separates the points of S and contains the constant mappings. The elements $\mathscr{P}(G) \otimes E$ are called continuous polynomials of finite type from G into E.

4. Simultaneous Approximation and Interpolation

We can apply our Theorem 1 to get results on simultaneous approximation and interpolation of *vector-valued* functions. Let us say that a subset $A \subset C(S; E)$ is an *interpolating family for* $C(S; E)$ if, given any finite subset $F \subset S$ and any $f \in C(S; E)$, there exists $g \in A$ such that $f(x) = g(x)$ for all $x \in F$.

THEOREM 6. *Let $A \subset C(S; E)$ be an interpolating family such that the set of multipliers of A separates the points of S. Then, for every $f \in C(S; E)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$. In particular, A is uniformly dense in $C(S; E)$.*

Proof. Define $W = \{g \in A;\ f(x) = g(x) \text{ for all } x \in F\}$. Since A is an interpolating family, $W \neq \varnothing$. Now it is easy to verify that each multiplier of A is also a multiplier of W. Hence, by Theorem 1, it suffices to show that, for each $x \in S$, there exists $g_x \in W$ such that $\|f(x) - g_x(x)\| < \varepsilon$. Consider the finite set $F \cup \{x\}$. Since A is an interpolating family for $C(S; E)$, there exists $g_x \in A$ such that $f(t) = g_x(t)$ for all $t \in F \cup \{x\}$. In particular, $f(t) = g_x(t)$ for all $t \in F$. Hence $g_x \in W$. On the other hand $f(x) = g_x(x)$ implies $\|f(x) - g_x(x)\| = 0 < \varepsilon$. By Theorem 1, there exists $g \in W$ such that $\|f - g\| < \varepsilon$, and $g \in W$ implies $g \in A$ and $g(t) = f(t)$ for all $t \in F$. ∎

Remark. When $E = \mathbb{K}$, then the conclusion of Theorem 6 is true under the hypothesis that $A \subset C(S; \mathbb{K})$ is a dense linear subspace. See

Deutsch [4]. This poses the question of finding dense linear subspaces of $C(S; E)$ for which the conclusion of Theorem 6 is valid, i.e., for which simultaneous approximation and interpolation is possible.

THEOREM 7. *Let $W \subset C(S; E)$ be a linear subspace such that*

$$A = \{\varphi \circ g;\ \varphi \in E^*,\ g \in W\}$$

is dense in $C(S; \mathbb{K})$ and $A \otimes E \subset W$. Then, for every $f \in C(S; E)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in W$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.

Proof. Case 1. $F = \emptyset$. By Theorem 5, the space $C(S; \mathbb{K}) \otimes E$ is dense in $C(S; E)$. Hence there is some $g \in C(S; \mathbb{K}) \otimes E$ such that $\|f - g\| < \varepsilon/2$. Let

$$g = \sum_{k=1}^{m} h_k v_k,$$

where $h_i \in C(S; \mathbb{K})$, $v_i \in E$, $i = 1, \dots, m$. Let $\varphi_1, \dots, \varphi_m \in E^*$ and $g_1, \dots, g_m \in W$ be such that $\|\varphi_i \circ g_i - h_i\| < \delta$, where $\delta > 0$ has been chosen so that $\delta \sum_{i=1}^{m} \|v_i\| < \varepsilon/2$. Then $w = \sum_{k=1}^{m} (\varphi_k \circ g_k) v_k$ belongs to $A \otimes E \subset W$ and $\|w - f\| < \varepsilon$.

Case 2. $F = \{x_1, \dots, x_n\} \neq \emptyset$. We first remark that A is an interpolating family for $C(S; \mathbb{K})$. Indeed, if we define $T\colon C(S; \mathbb{K}) \to \mathbb{K}^n$ by

$$Tg = (g(x_1), \dots, g(x_n))$$

for each $g \in C(S; \mathbb{K})$, then by density of A and continuity of T, we have

$$T(C(S; \mathbb{K})) = T(\bar{A}) \subset \overline{T(A)} = T(A),$$

where the last equality is a consequence of the fact that $T(A)$ is a linear subspace of $\mathbb{K}^n$, because A is a linear subspace of $C(S; \mathbb{K})$. Let $a_1, \dots, a_n \in A$ be such that

$$a_i(x_j) = \delta_{ij}, \qquad 1 \le i, j \le n.$$

Choose $\delta > 0$ so that $\delta(1 + \sum_{i=1}^{n} \|a_i\|) < \varepsilon$. By Case 1, W is dense in $C(S; E)$. Hence there is some $g_1 \in W$ such that $\|f - g_1\| < \delta$. Let

$$v_i = f(x_i) - g_1(x_i), \qquad 1 \le i \le n.$$

Since $A \otimes E \subset W$, it follows that

$$g_2(x) = \sum_{i=1}^{n} a_i(x) v_i, \qquad x \in S,$$

belongs to W. Notice that $g_2(x_j) = v_j$ for all $1 \leq j \leq n$. Hence $g(x_j) = f(x_j)$, for all $1 \leq j \leq n$, if $g \in W$ is defined to be $g_1 + g_2$. On the other hand,

$$\|f - g\| < \delta + \|g_2\| < \delta + \delta \sum_{i=1}^{n} \|a_i\| < \varepsilon. \quad \blacksquare$$

COROLLARY 4. *Let $W \subset C(S; E)$ be a dense linear subspace such that $A \otimes E \subset W$, where*

$$A = \{\varphi \circ g; \varphi \in E^*, g \in W\}.$$

Then, simultaneous approximation and interpolation from W is possible.

Proof. By Theorem 7, it suffices to show that A is dense in $C(S; \mathbb{K})$. Let $f \in C(S; \mathbb{K})$ and $\varepsilon > 0$ be given. Choose $v \in E$ and $\varphi \in E^*$ such that $\|\varphi\| \leq 1$ and $\varphi(v) = 1$. Let $g(x) = f(x)v$, for all $x \in S$. Then, by density of W, there is some $w \in W$ such that $\|w - g\| < \varepsilon$. Let $h = \varphi \circ w$. Then $h \in A$ and $|h(x) - f(x)| = |\varphi(w(x)) - f(x)\varphi(v)| = |\varphi(w(x)) - \varphi(f(x)v)| = |\varphi(w(x) - g(x))| \leq \|\varphi\| \cdot \|w(x) - g(x)\| < \varepsilon$, for all $x \in S$. $\blacksquare$

COROLLARY 5. *Let F and E be two real normed spaces, and let $S \subset F$ be a compact subset. Then, simultaneous approximation and interpolation is possible from the space of continuous polynomials of finite type from F into E.*

Proof. Let $W \subset C(S; E)$ be the restriction to S of the space $\mathscr{P}(F) \otimes E$. (For the definition of $\mathscr{P}(F)$ see Example 2.) Notice that $A = \{\varphi \circ g; \varphi \in E^*, g \in W\}$ is the restriction of $\mathscr{P}(F)$ to S. Hence $A \otimes E \subset W$. We saw in Example 3 that the polynomial algebra W is dense in $C(S; E)$. It remains to apply the previous corollary. $\blacksquare$

5. THE NON-SEPARATING CASE

A very simple modification of the proof of Lemma 3 yields the following *non-separating* version of it.

LEMMA 4. *Let $M \subset C((S; [0, 1])$ be a subset with property V. Let $x \in S$ and let N be an open neighborhood of $[x]_M$ in S. There exists an open* *neighborhood U of* $[x]_M$, *contained in N, such that, for each* $0 < \delta < 1/2$, *there is an element* $\varphi \in M$ *such that*

(1) $\varphi(t) > 1 - \delta$, *for all* $t \in U$,

(2) $\varphi(t) < \delta$, *for all* $t \notin N$.

In the statement of Lemma 4 we have used the following notation. For each $x \in S$, the set $[x]_M$ is the equivalence class of the point x for the following equivalence relation: $x \equiv y$ (mod. M) if, and only if, $\varphi(x) = \varphi(y)$, for all $\varphi \in M$. In the same way that Lemma 3 implies Theorem 1, Lemma 4 implies the following non-separating version of Theorem 1.

THEOREM 8. *Let W be a non-empty subset* $C(S; E)$, *and let M be the set of all multipliers of W. Let* $f \in C(S; E)$ *and* $\varepsilon > 0$ *be given. The following statements are equivalent:*

(1) *there is some* $g \in W$ *such that* $\|f - g\| < \varepsilon$;

(2) *for each* $x \in S$, *there is some* $g_x \in W$ *such that* $\|f(t) - g_x(t)\| < \varepsilon$, *for all* $t \in [x]_M$.

Analogously, in the same way that Theorem 2 follows from Theorem 1, the following result follows from Theorem 8.

THEOREM 9. *Let W be a non-empty subset of* $C(S; E)$, *and let M be the set of all multipliers of W. For each* $f \in C(S; E)$ *there is some* $x \in S$ *such that*

$$\operatorname{dist}(f; W) = \operatorname{dist}(f[x]_M; W[x]_M).$$

In the statement above we have used the following notation: $f[x]_M$ stands for the restriction of f to the compact subset $[x]_M \subset S$. Similarly, $W[x]_M$ denotes the set $\{g[x]_M;\ g \in W\}$ which is contained in the space $C([x]_M; E)$.

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